Ricci Flat Manifolds

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April 20, 2022

Pseudo-Riemannian Manifold

Definition (Pseudo-Riemannian Manifold)

A pseudo-Riemannian manifold is a pair (M, g) where M is a smooth manifold, and g is an everywhere non-degenerate, smooth, symmetric metric tensor.

Definition (Ricci Curvature/Ricci Flat Manifolds)

The Ricci Curvature for a pseudo-Riemannian manifold is

$$\text{Ric}_p = Tr(Z \to R(X, Z)Y)$$

If the Ricci curvature of a pseudo-Riemannian manifold (M, g) is zero we call (M, g) a **Ricci Flat Manifold**

Lorentzian Manifolds

Lorentzian Manifolds

A pseudo-Riemannian manifold (M, g) is called **Lorentzian** is the metric has signature (1,n-1)

Einstein's Field Equation

Let (M, g) be a 4-dimensional Ricci Flat Lorenzian Manifold.

Einstein's Field Equations

$$\mathrm{Ric} - \frac{1}{2} K g = T$$

Vacuum Solution

"A solution"

By a solution to Einsteins' equation we mean a metric g and a stress-energy tensor T.

Schwarzschild Solution

- A famous solution of the $\Lambda = 0$ Einstein equation $\mathrm{Ric} = 0$ is due to Schwarzschild in 1916
- Let $M = \mathbb{R} \times I \times S^2$ with any warped product metric

$$g = F^2(\rho)\, dt^2 - d\rho^2 - G^2(\rho)\, d\sigma^2$$

- Schwarzschild arrived at the following metric.

$$g = \left(1 - \frac{2m}{\tilde{\rho}}\right) dt^2 - \left(1 - \frac{2m}{\tilde{\rho}}\right)^{-1} d\tilde{\rho}^2 - \tilde{\rho}^2\, d\sigma^2$$

Flatness and Ricci Flatness

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A Riemannian Manifold (not necessarily pseudo) is called **flat** is the Riemann Curvature Tensor is 0

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Question

Do there exist Ricci Flat Manifolds which are not Flat?

Calabi Conjecture

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(Special case of) Calabi Conjecture

If the first Chern class vanishes there is a unique Kähler metric in the same class with vanishing Ricci curvature; these are called Calabi–Yau manifolds.

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Is every Ricci-flat Riemannian metric on a closed manifold flat?

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Theorem (Calabi-Yau)

These exist for Ricci-flat metrics in the special case of Kähler metrics on closed complex manifolds.

Resources

- Besse, A L. Einstein Manifolds. Berlin: Springer-Verlag, 1987. Print.
- Carmo, Manfredo P. Riemannian Geometry. Boston: Birkhauser, 1993. Print.
- O'Neill, Barrett. Semi-riemannian Geometry with Applications to Relativity. Burlington: Elsevier, 1983.